Announcements

1. Quiz 2 available due 7/1 11:59 PM
2. PS 3 due date has been pushed to 7/6
3. Assignments calendar up to midterm has been published
4. PS 4 will be uploaded by 7/1 at noon

§ 2.5 Continued

<u>Note</u>:

- We can determine the concavity of curves using

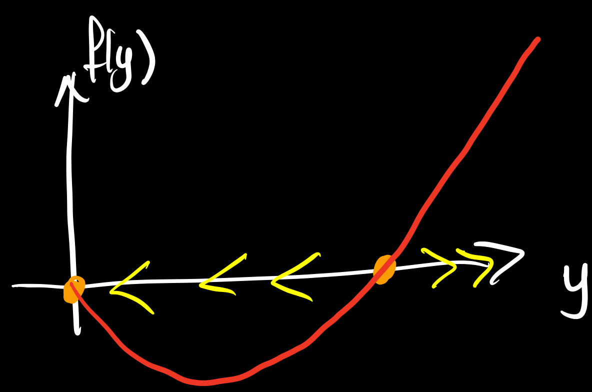
$$\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt} f(y) = f'(y)\frac{dy}{dt}$$
$$= f'(y) f(y)$$

- Solutions tend to K as $t \to \infty$
  K is called the saturation level or carrying capacity
- $y = K$ is called a stable equilibrium
  $y = 0$ is called an unstable equilibrium

Critical Threshold

Consider

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y$$



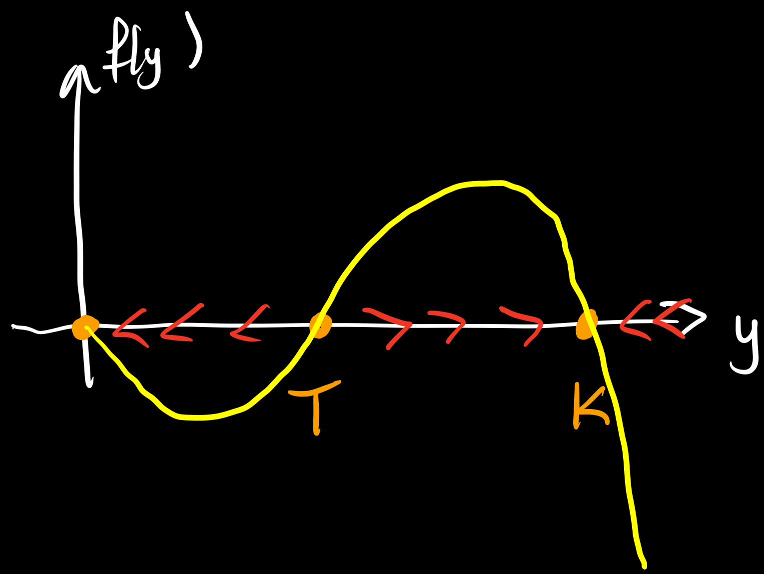
T is called the threshold level

$y > T \Rightarrow$ growth

$y < T \Rightarrow$ decay

Logistic Growth with a Threshold

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y \,, \quad T<K$$



§2.6 Exact DEs and Integrating Factors

Consider the DE

$$M(x,y) + N(x,y)\, y' = 0 .$$

This DE is not necessarily linear nor separable, so we do not yet have an applicable solution method.

Suppose we could find a function $\psi(x,y)$

such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

and $\psi(x,y) = c$ defines $y = \varphi(x)$ implicitly. Now,

$$M(x,y) + N(x,y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} y'$$

$$\boxed{\frac{d}{dx}\psi(x,y) = \frac{\partial \psi}{\partial x}\frac{dx}{dx} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}}$$

$$= \frac{d}{dx}\psi(x,y)$$

$$= \frac{d}{dx}\psi(x,\varphi(x))$$

and so the DE becomes

$$\frac{d}{dx}\psi(x,\varphi(x)) = 0.$$

Such a DE is called an exact differential equation.

How can we determine whether a DE is exact and how do we find $\psi(x,y)$?

<u>Thm</u> Let M, N, M_y, and N_x be continuous in the region $R: \alpha < t < \beta$, $\gamma < y < \delta$. Then

$$M(x,y) + N(x,y)\, y' = 0$$

is an exact DE in R if and only if

$$M_y(x,y) = N_x(x,y)$$

at each point of R.

<u>Remark</u> We find ψ by solving

$$\psi_x = M \quad \text{and} \quad \psi_y = N.$$

Ex $y\cos x + 2xe^y + (\sin x + x^2e^y - 1)y' = 0$

$$\frac{\partial}{\partial y} M = \frac{\partial}{\partial y}\left(y\cos x + 2xe^y\right)$$
$$= \cos x + 2xe^y$$

$$\frac{\partial}{\partial x} N = \frac{\partial}{\partial x}\left(\sin x + x^2e^y - 1\right)$$
$$= \cos x + 2xe^y$$

So $\quad M_y = N_x$

So the DE is exact. Now, we want to find $\psi(x,y)$. Such a function ψ satisfies

$$\psi_x = y\cos x + 2xe^y$$

$$\psi_y = \underline{\sin x + x^2 e^y - 1}$$

Now,

$$\int \psi_x \, dx = \int y\cos x + 2xe^y \, dx$$

$$\psi = \underline{y\sin x + x^2 e^y + h(y)}$$

Next, we know ψ_y so

$$\sin x + x^2 e^y - 1 = \frac{\partial}{\partial y}\left(y\sin x + x^2 e^y + h(y)\right)$$

$$= \sin x + x^2 e^y + h'(y)$$

$$-1 = h'(y)$$

$$-y + C = h(y)$$

Therefore,

$$\psi(x,y) = y\sin x + x^2 e^y - y + C.$$

or

$$C = y\sin x + x^2 e^y - y$$

<u>Integrating Factors</u>

It is sometimes possible to convert a DE that is not exact into an exact DE by multiplying by an integrating factor $\mu(x,y)$. Then we have

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

and for the DE to be exact, we must have

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu_y M - \mu_x N = \mu(N_x - M_y)$$

M and N are known, but we have a PDE for $\mu(x,y)$

$$\frac{\partial \mu}{\partial y} M - \frac{\partial \mu}{\partial x} N = \mu(N_x - M_y).$$

This PDE may have more than one solution, but we only need one of them to use as an integrating factor.

We can make the assumption that μ depends only on x (or y). Then the PDE becomes an ODE

$$-\frac{d\mu}{dx} N = \mu (N_x - M_y)$$

Since $\mu = \mu(x)$ and so $\frac{\partial}{\partial y} \mu = 0$.

The ODE is linear and separable, so we can find μ!

<u>Ex</u> Find an integrating factor for

$$3xy + y^2 + (x^2 + xy) y' = 0$$

Let's try $\mu = \mu(x)$. Then

$$-\frac{d\mu}{dx}(x^2+xy) = \mu\left(\frac{\partial}{\partial x}(x^2+xy) - \frac{\partial}{\partial y}(3xy+y^2)\right)$$

$$-\frac{d\mu}{dx}(x^2+xy) = \mu(2x+y-3x-2y)$$

$$+\frac{d\mu}{dx} = \frac{\mu(+x+y)}{x^2+xy}$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{x}$$

$$\int \frac{d}{dx}(\ln\mu)\,dx = \int \frac{1}{x}\,dx$$

$$\ln\mu = \ln x$$

$$\mu = x$$

§2.7 Numerical Approximations: Euler's Method

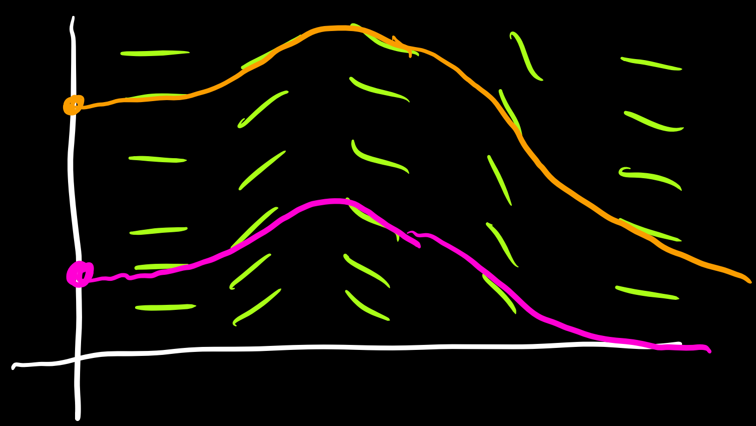
We have seen a few examples of 1st order DE that we can solve symbolically, but more often than not, we cannot solve the DE "by hand". We know that the IVP

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

has a unique solution $y = \phi(t)$ if f and $\frac{\partial f}{\partial y}$ are continuous. How can we "see" the solution?

One method is to draw a direction field and sketch some solution curves. This gives us good qualitative results, but we may want more concrete qualitative results.

What are we doing when we draw in solution curves?



$\longrightarrow$ Tangent line method or Euler method

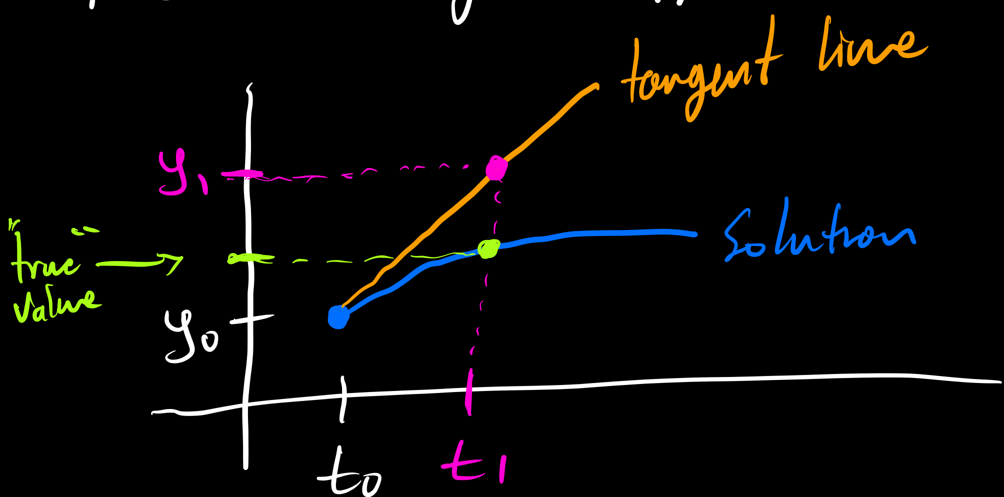
Consider the IVP

$$\frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0 .$$

The solution passes through (t_0, y_0) and the slope at (t_0, y_0) is $f(t_0, y_0)$ so the tangent line to the solution at (t_0, y_0) is

$$y = y_0 + f(t_0, y_0)(t - t_0).$$

As long as we stay close to $t = t_0$, this is a good approximation.



Next, we want a tangent line

approximation at $t=t_1$, but we do not know the solution at t_1!

Let's use the tangent line at (t_0, y_0) to approximate y_1, so

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$

Next, we want to find (t_2, y_2), so using the tangent line at (t_1, y_1), we have

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

and so

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$$

and we get a sequence $y_0, y_1, \dots$ at $t_0, t_1, \dots$ that approximates the solution.

<u>Note</u> If we assume a uniform step size h between $t_0, t_1, t_2, \dots$, then $t_{n+1} = t_n + h$, so

$$y_{n+1} = y_n + h\, f(t_n, y_n).$$

<u>Remark</u> Euler's method is a sequence of tedious computations, so rather than using this method by hand, we will usually use a computer.

The numerical method above is called

(forward/explicit) Euler. There is also what's called backward/implicit Euler. Instead of evaluating $f(t,y)$ at the current value (t_n, y_n), we evaluate f at (t_{n+1}, y_{n+1}). Then our method becomes

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}).$$

For a specific function f, we rearrange the equation so that we get $y_{n+1} = \cdots$.

<u>Ex</u> Set up the Euler iteration scheme for IVP

$$\frac{dy}{dt} = t^2 + 5y \quad , \quad y(0) = 1 .$$

Explicit :
$$y_{n+1} = y_n + h\,f(t_n, y_n)$$
$$= y_n + h\,(t_n^2 + 5y_n)$$
$$= y_n(1+5h) + h\,t_n^2$$

Implicit :
$$y_{n+1} = y_n + h\,f(t_{n+1}, y_{n+1})$$
$$= y_n + h\,(t_{n+1}^2 + 5y_{n+1})$$

$$y_{n+1}(1-5h) = y_n + h\,t_{n+1}^2$$

$$y_{n+1} = \frac{y_n + h t_{n+1}^2}{1-5h}$$

$$= \frac{y_n + h\,(t_n+h)^2}{1-5h}$$